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CAUCHY INTEGRAL METHOD FOR TWO-DIMENSIONAL SOLIDIFICATION INTERFACE SHAPES

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Abstract—A method is developed to determine the shape of steady state solidification interfaces formed when liquid above its freezing point circulates over a cold surface. The solidification interface, which is at uniform temperature, will form in a shape such that the non-uniform energy convected to it is locally balanced by conduction into the solid. The interface shape is of interest relative to the crystal structure formed during solidification; regulating the crystal structure has application in casting naturally strengthened metallic composites. The results also pertain to phase-change energy storage devices, where the solidified configuration and overall heat transfer are needed. The analysis uses a conformal mapping technique to relate the desired interface coordinates to the components of the temperature gradient at the interface. These components are unknown because the interface shape is unknown. A Cauchy integral formulation provides a second relation involving the components, and a simultaneous solution yields the interface shape.

NOMENCLATURE

b, height of region, $B = b/\gamma$;
C, constant of integration;
d, $= (1 - k^2)/k^2$, location on ξ-axis of u-plane;
F, function specifying variation of q along interface;
$F(\phi, k)$, elliptic integral of the first kind;
G, $= q_s(s)/q_r$, ratio of local heat flux along interface to reference flux;
$K(k)$, complete elliptic integral of the first kind;
k_m, thermal conductivity of solid;
k, modulus of elliptic integral, $k' = \sqrt{1 - k^2}$;
n, normal to interface, $N = n/\gamma$;
Q, total heat flow rate through solid region, $Q = Q/k_m(t_s - t_w)$;
q, heat flow rate per unit area, $\tilde{q} = q\gamma/k_m(t_s - t_w) = q/\bar{q}$;
$\bar{q}$, $= Q/b$, average heat flux through region;
Δq, amplitude of imposed heat flux variation along interface;
s, coordinate along solidification interface, $S = s/\gamma$;
T, $= (t - t_w)/(t_s - t_w)$, temperature ratio;
t, temperature;
U, V, integrals defined in equations (39) and (40);
u, $= \xi + i\eta$, intermediate mapping plane;
W, $= -T + i\psi$, complex potential function;
x, y, Cartesian coordinates, $X = x/\gamma$, $Y = y/\gamma$;
z, $= x + iy$, complex variable;
Z, $= z/\gamma$;
∇^2, $= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$;
$\tilde{\nabla}^2$, $= \dfrac{\partial^2}{\partial X^2} + \dfrac{\partial^2}{\partial Y^2}$.

Greek symbols

γ, $= k_m(t_s - t_w)b/Q = k_m(t_s - t_w)/\bar{q}$, length scale quantity;
η, ordinate in u-plane;
θ, angle from x direction to normal of interface, see Fig. 1;
ξ, abscissa in u-plane;
$\bar{\xi}$ dummy variable of integration;
ψ, ordinate in W-plane.

Subscripts

r, reference value when $\Delta q = 0$;
s, at solidification interface;
w, cooled wall;
1, 2, 3, 4, the four corners of the solidified region, see Fig. 1.

Superscripts

A, B, refer to Quad A and Quad B integration rules.

INTRODUCTION

IN A HEAT conduction analysis the geometry of the solid is usually known, and various boundary conditions are applied to yield the temperature distribution or heat flows within the body. The present analysis will deal with a situation where an additional boundary condition is specified. The shape of the body is unknown and is to be found to satisfy this additional constraint. This type of analysis has application for obtaining equilibrium shapes of a solid in contact with its superheated liquid phase. The interface between the phases is at the solidification temperature and also has heat transfer to it from the liquid phase. The interface will form such that the convective heat transfer to it can be conducted away by the solid to the portion of the boundary that is being cooled. The analysis will show how the interface shape is influenced by non-uniformities in the convective heat transfer.

The present method is pertinent to the analysis of continuous casting processes for manufacturing strengthened metallic components. An analysis of interface shapes during casting was made in [1] for uniform convection at the solidification interface. By having convective conditions that are sufficiently well regulated at the interface, a strengthened composite structure can be formed consisting of a reinforcing phase within a metallic matrix. Related situations are the effects of natural convection in the melted region around a cylinder [2], the shape of a steady state frozen layer formed on a cold plate within a warm liquid flow [3, 4], and the shapes of solidified regions in phase change energy storage devices. The analytical results can also be used for comparison with solutions by finite difference procedures.

This analysis considers a solid region that has one boundary at a temperature below the freezing point. Another boundary is the solidification interface in contact with the liquid phase. The interface shape is found subject to constraints of uniform temperature and an imposed heat flux distribution. A conformal mapping procedure developed in [5] can be used to obtain the interface coordinates if the relation is known between components of the temperature gradient along the interface; this relation is found by using a Cauchy integral formulation. To demonstrate the method and provide useful results, solidified shapes are obtained for a cosine heating variation along the interface. Analytical results were found in [6] for small cosine heating amplitudes, and good agreement is obtained. Other heating distributions can be used as obtained by coupling the present solution with an analysis in the liquid phase. The use of conformal mapping limits the method to 2-dim. configurations.

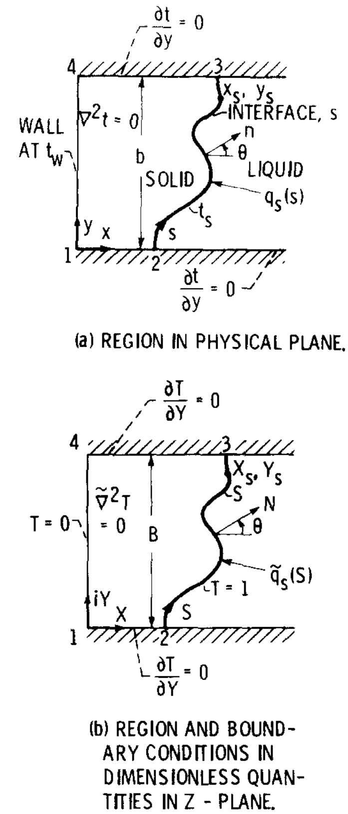


FIG. 1. Solidified region formed as a result of non-uniform heating along surface.

ANALYTICAL FORMULATION

The geometry is shown in Fig. 1(a). A container is bounded on two sides by insulated walls 1–2 and 3–4, a distance b apart. Since there is zero heat flow through these walls:

$$\frac{\partial t}{\partial y} = 0, \quad y = 0 \quad \text{and } b, \; x \geq 0. \tag{1}$$

The container is partially filled with warm liquid, and the remainder is solid that has frozen into an unknown shape to be determined by the analysis. The freezing is caused by wall 1–4 being maintained at uniform temperature t_w below the freezing temperature t_s:

$$t = t_w < t_s, \quad x = 0, \quad 0 \leq y \leq b. \tag{2}$$

The liquid is at a temperature above t_s, and heat convection or conduction will provide energy transfer to the interface s. Since the heat transfer coefficient and temperature can vary within the liquid, the energy $q(s)$ supplied along the unknown interface is non-uniform. If n is the direction of the outward normal, then

$$q_s(s) = k_m \left.\frac{\partial t}{\partial n}\right|_s \quad x, y \text{ on } s. \tag{3}$$

The interface temperature is also uniform at the liquid solidification temperature:

$$t = t_s, \quad x, y \text{ on } s. \tag{4}$$

Within the solid, the temperature distribution must satisfy the heat conduction equation (constant properties are assumed),

$$\nabla^2 t = 0. \tag{5}$$

The shape of the interface will adjust until the heat conduction within the solid can transport away to the cooled wall the energy $q_s(s)$ supplied to the interface. This must be done under the constraint that s remains at uniform temperature, t_s.

The analysis is partially based on a conformal mapping technique for free boundary solidification in [5]. It is more convenient to work with dimensionless variables so $T = (t - t_w)/(t_s - t_w)$ is used that varies between 0 and 1 in the solid. All lengths are divided by a characteristic length γ defined later. The dimensionless heat flux is $\tilde{q} \equiv q\gamma/k_m(t_s - t_w)$. Then equations (1)–(5) become:

$$\frac{\partial T}{\partial Y} = 0, \quad Y = 0, B, \quad X \geq 0 \tag{6}$$

$$T = 0, \quad X = 0, \quad 0 \leq Y \leq B \tag{7}$$

$$\tilde{q}_s(S) = \left.\frac{\partial T}{\partial N}\right|_S \quad X, Y \text{ on } S \tag{8}$$

$$T = 1 \quad X, Y \text{ on } S \tag{9}$$

$$\tilde{\nabla}^2 T = 0. \tag{10}$$

These conditions are summarized in Fig. 1(b).

Solid region in W- *and* u-*planes*

The quantity $-T$ can be regarded as the potential function for dimensionless heat flow, because the derivative of $-T$ in a direction is equal to the dimensionless heat flux in that direction. An analytic function W of a complex variable Z is defined as

$$W \equiv -T + \mathrm{i}\psi. \tag{11}$$

The W, T and ψ all satisfy Laplace's equation, and lines of constant T and ψ form an orthogonal curvilinear net in the physical plane. The heat flow, being normal to the constant T lines, is along constant ψ lines.

In the W-plane the solidified region occupies a rectangle of unit width (Fig. 2). To obtain the height of the rectangle consider the total heat flow Q through the solid region. Integrating along the boundary 1–4,

$$Q = k_m \int_{y_1}^{y_4} (\partial t/\partial x)\, \mathrm{d}y,$$

or in dimensionless form,

$$\tilde{Q} = \int_{Y_1}^{Y_4} (\partial T/\partial X)\, \mathrm{d}Y.$$

From the Cauchy–Riemann equations, $\partial T/\partial X = -\partial\psi/\partial Y$, so integration yields $\tilde{Q} = \psi_1 - \psi_4$ as shown in Fig. 2.

Now consider the characteristic length γ. It is convenient to let γ be an average thickness in the x-direction of the solidified region. In the limit when the region has uniform thickness x_s, then γ equals x_s and the average heat flux at the interface is $Q/b = k_m(t_s - t_w)/\gamma$. Thus a convenient characteristic thickness is $\gamma \equiv k_m(t_s - t_w)b/Q = b/\tilde{Q}$. It follows that $B = b/\gamma = \tilde{Q}$, and the dimensionless thickness of the uniform region is $x_s/\gamma = 1$. The regions in Figs. 1(b) and 2 have the same height and about the same average width, which will aid in the conformal mapping between them that will be required. For a given $q_s(s)$ variation, the Q used in γ is an unknown since it involves an integration of $q_s(s)$ over the unknown interface s. In the solution, the relation between $q_s(s)$ and Q will be found so that quantities containing Q can then be expressed in terms of the known $q_s(s)$.

The derivative of W is from equation (11), $\partial W/\partial Z = -\partial T/\partial X + \mathrm{i}\partial\psi/\partial X$. Using the Cauchy–Riemann equation $\partial\psi/\partial X = \partial T/\partial Y$, the ψ is eliminated so that

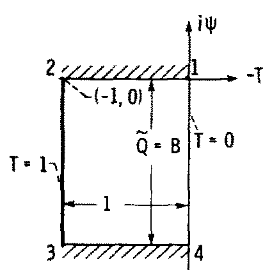


FIG. 2. Solidified region mapped into potential plane, $W = -T + \mathrm{i}\psi$.

$$\frac{\partial W}{\partial Z} = -\frac{\partial T}{\partial X} + \mathrm{i}\frac{\partial T}{\partial Y}. \tag{12}$$

Equation (12) is rearranged and integrated to yield

$$Z = \int \frac{\mathrm{d}W}{-\dfrac{\partial T}{\partial X} + \mathrm{i}\dfrac{\partial T}{\partial Y}} + C. \tag{13}$$

If $\partial T/\partial X$ and $\partial T/\partial Y$ can be related to W, the integration can be carried out to obtain Z as a function of W. Since the interface 2–3 is a known straight line in the W-plane (Fig. 2), the interface in the physical plane can then be found.

The relation between the temperature derivatives and W is found by using an intermediate u-plane shown in Fig. 3. The unknown interface is in the convenient interval, $-1 \leq \xi \leq 0$. The mapping between u and W is obtained from a Schwarz–Christoffel transformation:

$$\frac{\mathrm{d}W}{\mathrm{d}u} = \frac{1}{2K(k')} \frac{1}{\sqrt{u+1}\sqrt{u}\sqrt{1-k^2(1+u)}}, \tag{14}$$

as shown in the Appendix. The k is found from equation (A.3),

$$\tilde{Q} = \frac{K(k)}{K(k')} \tag{15}$$

and the point d in Fig. 3 is at $d = (1-k^2)/k^2$. Then equation (13) can be written as [by letting $\mathrm{d}W = (\mathrm{d}W/\mathrm{d}u)\mathrm{d}u$]:

$$Z = \frac{1}{2K(k')} \times \int \frac{\mathrm{d}u}{\left(-\dfrac{\partial T}{\partial X} + \mathrm{i}\dfrac{\partial T}{\partial Y}\right)\sqrt{u+1}\sqrt{u}\sqrt{1-k^2(1+u)}} + C. \tag{16}$$

What is now needed is to relate the temperature derivatives to u.

Cauchy integral for temperature derivatives at interface

At point 1, which is at $\xi \rightarrow \pm\infty$ in Fig. 3, the derivative $\partial T/\partial X \neq 0$ because there is heat flow in the

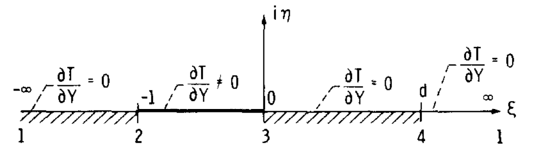


FIG. 3. Solidified region mapped into upper half of u-plane with interface in region $-1 \leqslant \xi \leqslant 0$.

X direction throughout the solidified region. However the quantity $(\partial T/\partial X) - (\partial T/\partial X)|_1 \to 0$ as $\xi \to \pm\infty$. With this condition satisfied, the relations on p. 372 of [7] can be used to relate $(\partial T/\partial X) - (\partial T/\partial X)|_1$ to $\partial T/\partial Y$ along the real axis of the u-plane:

$$-\frac{\partial T}{\partial X}(\xi) + \frac{\partial T}{\partial X}\bigg|_1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\frac{\partial T}{\partial Y}(\bar{\xi})}{\bar{\xi} - \xi}\,d\bar{\xi}. \quad (17)$$

From equations (6) and (7) the $\partial T/\partial Y = 0$ along boundaries 1–2, 3–4 and 4–1, and $\partial T/\partial Y$ is non-zero only along 2–3 as a result of equation (8). Then equation (17) can be simplified so that the integration extends only between $\xi = -1$ and 0;

$$-\frac{\partial T}{\partial X}(\xi) + \frac{\partial T}{\partial X}\bigg|_1 = \frac{1}{\pi}\int_{\bar{\xi}=-1}^{0} \frac{\frac{\partial T}{\partial Y}(\bar{\xi})}{\bar{\xi} - \xi}\,d\bar{\xi}. \quad (18)$$

Since equation (18) will eventually be solved along 2–3 the $(\partial T/\partial X)|_1$ will be eliminated in favor of $(\partial T/\partial X)|_2$ which is at an end point of 2–3. Equation (18) evaluated at $\xi = -1$ (point 2) gives

$$-\frac{\partial T}{\partial X}\bigg|_2 + \frac{\partial T}{\partial X}\bigg|_1 = \frac{1}{\pi}\int_{\bar{\xi}=-1}^{0} \frac{\frac{\partial T}{\partial Y}(\bar{\xi})}{\bar{\xi} + 1}\,d\bar{\xi}. \quad (19)$$

Subtracting equation (19) from (18) to yield, after combining terms under the integral,

$$-\frac{\partial T}{\partial X}(\xi) + \frac{\partial T}{\partial X}\bigg|_2 = \frac{\xi + 1}{\pi}\int_{\bar{\xi}=-1}^{0} \frac{\frac{\partial T}{\partial Y}(\bar{\xi})}{(\bar{\xi} - \xi)(\bar{\xi} + 1)}\,d\bar{\xi}. \quad (20)$$

The heat flux $q_s(s)$ imposed at the interface in Fig. 1 will be written as

$$q_s(s) = q_r + \Delta q\, F(s) \quad (21)$$

where q_r is a reference value, Δq is an amplitude, and for some applications $F(s)$ could be found by coupling this analysis with a solution for heat transfer in the liquid. Then

$$\frac{q_s(S)}{q_r} = 1 + \frac{\Delta q}{q_r}F(S) \equiv G[S(\xi)] = G(\xi). \quad (22)$$

If an average heat flux through the region is defined as $\bar{q} \equiv Q/b$ then $\gamma = k_m(t_s - t_w)/\bar{q}$ and the dimensionless heat flux $\tilde{q}_s(S) = q_s(S)/\bar{q}$. The boundary condition, equation (8), becomes

$$\frac{\partial T}{\partial N}\bigg|_S = \frac{q_s(S)}{\bar{q}} = \frac{q_r}{\bar{q}}G[S(\xi)]. \quad (23)$$

The q_r and G are known from the specified equation (21). However $\bar{q}$ is not known, and the ratio $\bar{q}/q_r$ will be found in the solution.

The normal derivative (23) at the constant temperature interface can be resolved into two components

$$\frac{\partial T}{\partial X}\bigg|_S = \frac{q_r}{\bar{q}}G(S)\cos\theta(S) \quad (24a)$$

$$\frac{\partial T}{\partial Y}\bigg|_S = \frac{q_r}{\bar{q}}G(S)\sin\theta(S). \quad (24b)$$

These are substituted into equation (20) to yield along the interface,

$$G(\xi)\cos\theta(\xi) = G(\xi = -1) - \frac{\xi + 1}{\pi} \times \int_{\bar{\xi}=-1}^{0} \frac{G(\bar{\xi})\sin\theta(\bar{\xi})}{(\bar{\xi} - \xi)(\bar{\xi} + 1)}\,d\bar{\xi}, \quad -1 \le \xi \le 0. \quad (25)$$

This Cauchy integral equation will be solved later to obtain the angle θ along the interface which is in the interval $-1 \le \xi \le 0$.

Relations for interface coordinates

To develop equations for the interface coordinates, equations (24) are substituted into equation (16), to yield after rearrangement,

$$Z_s(\xi) - Z_2 = \frac{1}{2K(k')}\frac{\bar{q}}{q_r} \times \int_{\bar{\xi}=-1}^{\xi} \frac{-\sin\theta(\bar{\xi}) + i\cos\theta(\bar{\xi})}{G(\bar{\xi})} \times \frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1 - k^2(1+\bar{\xi})}}. \quad (26)$$

The dimensionless height B equals $\mathrm{Im}[Z_s(\xi = 0) - Z_2]$; using this in (26) gives a relation for $q_r/\bar{q}$:

$$\frac{q_r}{\bar{q}} = \frac{1}{2BK(k')} \times \int_{\bar{\xi}=-1}^{0} \frac{\cos\theta(\bar{\xi})}{G(\bar{\xi})} \times \frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1 - k^2(1+\bar{\xi})}}. \quad (27)$$

The imaginary part of equation (26) divided by B from (27) yields the normalized Y-coordinate along the interface as

$$\frac{Y_s(\xi)}{B} = \frac{\displaystyle\int_{\bar{\xi}=-1}^{\xi} \frac{\cos\theta(\bar{\xi})}{G(\bar{\xi})}\frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1 - k^2(1+\bar{\xi})}}}{\displaystyle\int_{\bar{\xi}=-1}^{0} \frac{\cos\theta(\bar{\xi})}{G(\bar{\xi})}\frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1 - k^2(1+\bar{\xi})}}}. \quad (28)$$

The real part of equation (26) gives the X-coordinate along the interface relative to X_2 as

$$X_s(\xi) - X_2 = -\frac{1}{2K(k')}\frac{\bar{q}}{q_r} \times \int_{\bar{\xi}=-1}^{\xi} \frac{\sin\theta(\bar{\xi})}{G(\bar{\xi})} \frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1-k^2(1+\bar{\xi})}}. \quad (29)$$

The quantity $X_3 - X_2$ is found by letting $\xi = 0$ in equation (29).

Equations (28) and (29) give the shape of the interface, but to position the interface the X_3 or X_2 is needed. By integrating the real part of equation (16) between points 4 and 3 the X_3 is found as [note that $d = (1 - k^2)/k^2$]

$$X_3 = \frac{1}{2K(k')} \int_{\xi=0}^{(1-k^2)/k^2} \times \frac{1}{\frac{\partial T}{\partial X}(\xi)} \frac{d\xi}{\sqrt{\xi+1}\sqrt{\xi}\sqrt{1-k^2(1+\xi)}}. \quad (30)$$

To carry out this integral, $\partial T/\partial X$ is needed in the range $0 \le \xi \le (1 - k^2)/k^2$. From the forms of equation (20) and (24),

$$\frac{\partial T}{\partial X}(\xi) = \frac{q_r}{\bar{q}}\left[G(\xi = -1) - \frac{\xi+1}{\pi} \times \int_{\bar{\xi}=-1}^{0} \frac{G(\bar{\xi})\sin\theta(\bar{\xi})}{(\bar{\xi}-\xi)(\bar{\xi}+1)} d\bar{\xi}\right],$$

$$0 \le \xi \le \frac{1-k^2}{k^2}, -1 \le \bar{\xi} \le 0. \quad (31)$$

Since $\partial T/\partial X$ is needed along 3–4 it would be better if equation (31) contained $G(\xi = 0)$ corresponding to point 3 rather than $G(\xi = -1)$ corresponding to point 2. To accomplish this equation (25) is evaluated at point 3 ($\xi = 0$) to obtain

$$G(\xi=0) = G(\xi=-1) - \frac{1}{\pi}\int_{\bar{\xi}=-1}^{0} \frac{G(\bar{\xi})\sin\theta(\bar{\xi})}{\bar{\xi}(\bar{\xi}+1)} d\bar{\xi}.$$

Multiply this by $q_r/\bar{q}$ and subtract from equation (31). After simplification, this results in

$$\frac{\partial T}{\partial X}(\xi) = \frac{q_r}{\bar{q}}\left[G(\xi = 0) - \frac{\xi}{\pi}\int_{\bar{\xi}=-1}^{0} \frac{G(\bar{\xi})\sin\theta(\bar{\xi})}{\bar{\xi}(\bar{\xi}-\xi)} d\bar{\xi}\right],$$

$$0 \le \xi \le \frac{1-k^2}{k^2}, -1 \le \bar{\xi} \le 0. \quad (32)$$

The required relations have now been obtained, and the general procedure for using them will be outlined.

OUTLINE OF ITERATION PROCEDURE

In general a family of solutions is desired for various Δq amplitudes in equation (21) for a given heating function $F(s)$. Although the derivation is for a general $F(s)$, a specialization will be made to yield some illustrative results of physical interest, while somewhat shortening the numerical evaluation. In a confined geometry such as in Fig. 1, the heat transfer to the interface is governed by a cellular convective motion and is adequately specified as a function of vertical location. Then

$$G = 1 + \frac{\Delta q}{q_r} F(Y) \quad (33)$$

and solutions are found for various $\Delta q/q_r$ for an $F(Y)$ that is chosen.

Each solution will involve an iteration process during which the size of the rectangle in Fig. 2 is kept fixed. The first step is to choose a value for B, the dimensionless height in Fig. 2. Since the average dimensionless width of the solidified region is about unity, typical values of B were selected from 0.5–3.0. With $B = \tilde{Q}$ fixed, the values of k and k' are found from equation (15) and remain constant throughout the numerical evaluation for each case.

If $\Delta q/q_r = 0$, the solidification interface is a straight line $X_s = 1, 0 \le Y_s \le B$, and in equation (28), $\cos\theta = 1$ and $G = 1$. Then

$$\frac{Y_s(\xi)}{B} = \frac{\int_{\bar{\xi}=-1}^{\xi} \frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1-k^2(1+\bar{\xi})}}}{\int_{\bar{\xi}=-1}^{0} \frac{d\bar{\xi}}{\sqrt{\bar{\xi}+1}\sqrt{-\bar{\xi}}\sqrt{1-k^2(1+\bar{\xi})}}} = \frac{F(\phi, k)}{K(k)} \quad (34)$$

where $\phi = \sin^{-1}\sqrt{\xi+1}$ (see [8]). This gives a first approximation for the relation between Y_s and ξ.

Now let $\Delta q/q_r$ have a small value such as 0.1. Using equation (34) as a first approximation for Y_s as a function of ξ, find $G(\xi)$ from

$$G(\xi) = 1 + \frac{\Delta q}{q_r} F[Y_s(\xi)]. \quad (35)$$

Equation (25) is then solved for the angle $\theta(\xi)$ by a numerical procedure given later. These first approximations for $\theta(\xi)$ and $G(\xi)$ are used in equation (28) to find a second approximation for $Y_s(\xi)/B$. Keeping $\Delta q/q_r$ fixed, the new $Y_s(\xi)$ is used in equation (35) to obtain an improved $G(\xi)$ and equation (25) is solved again for an improved $\theta(\xi)$. The process is continued until $Y_s(\xi)/B$ and $\theta(\xi)$ converge. The ratio $q_r/\bar{q}$ is then found from equation (27). The $X_s(\xi) - X_2$ is found from equation (29), and when $\xi = 0$, this yields $X_3 - X_2$. The $\partial T/\partial X(\xi)$ is found from equation (32) and X_3 is obtained from equation (30). The interface is now known, since X_s and Y_s are both known parametrically in terms of ξ.

The above quantities are dimensionless, and they are now related to physical values. Relative to the physical height b, the interface is given by $x_s/b = X_s/B$ and $y_s/b = Y_s/B$. The total heat flow is given by $Q/k_m(t_s - t_w) = B$. The height b is related to dimensionless parameters by $bq_r/k_m(t_s - t_w) = B(q_r/\bar{q})$ where $q_r/\bar{q}$ was found in the analysis. For $\Delta q = 0$ the uniform thickness of the region in the x direction is

$k_m(t_s - t_w)/q_r$. The ratio of x_s to this thickness is $x_s q_r/k_m(t_s - t_w) = X_s(q_r/\bar{q})$. For a given $F(Y)$, a family of solutions is generated by choosing various B values and a series of $\Delta q/q_r$ for each B. The results are related to physical quantities by use of the above relations.

METHOD OF SOLUTION FOR CAUCHY INTEGRAL EQUATION

The Cauchy integral equation (25) was solved by writing it in a non-singular form and then employing an interspersed quadrature method similar to that described in [9]. Interspersing two integration schemes avoids the numerical difficulty of having the integration variable and the unknown variable at the same point. This would produce a zero in the denominator of one term in equation (25). The quadratures express the integral equation as a system of transcendental equations. These are then solved by a non-linear search routine through the minimization of the sum of squares of residuals as will be defined later.

Starting from equation (25), the relation $(\xi + 1)/(\bar{\xi} - \xi)(\bar{\xi} + 1) = 1/(\bar{\xi} - \xi) - 1/(\bar{\xi} + 1)$ is used to split the integral into two parts. Then in the first part, subtract and add a $G(\xi)\sin\theta(\xi)$ term to remove the singularity from the integral. The result is the non-singular form of the equation:

$$G(\xi)\left[\cos\theta(\xi) - \frac{1}{\pi}\sin\theta(\xi)\ln\left(\frac{\xi+1}{-\xi}\right)\right]$$

$$= G(\xi = -1) - \frac{1}{\pi}\int_{\bar{\xi}=-1}^{0} \frac{G(\bar{\xi})\sin\theta(\bar{\xi}) - G(\xi)\sin\theta(\xi)}{\bar{\xi} - \xi}\,\mathrm{d}\bar{\xi} + \frac{1}{\pi}\int_{\bar{\xi}=-1}^{0}\frac{G(\bar{\xi})\sin\theta(\bar{\xi})}{\bar{\xi}+1}\,\mathrm{d}\bar{\xi},$$

$$-1 \le \xi \le 0, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}. \quad (36)$$

Following [9] in a general manner, two quadrature rules, say Quad A and Quad B are employed. In general these can be represented as

$$\text{Quad A:} \quad \int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=1}^{N} w_k^A f(x_k^A) \quad (37)$$

$$\text{Quad B:} \quad \int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{N} w_k^B f(x_k^B) \quad (38)$$

where w_k^A and x_k^A represent the appropriate weights and integration nodal points for quadrature rule A, while w_k^B and x_k^B are the same for B. It is required that $a < x_k^A < b, k = 1, 2, \ldots N$, and $a \le x_k^B \le b, k = 0, 1, 2, \ldots N$ so that $x_0^B = a$, $x_N^B = b$.

Now apply these relations to the integrals in (36),

$$U(\xi) \equiv \int_{\bar{\xi}=-1}^{0} \frac{G(\bar{\xi})\sin\theta(\bar{\xi}) - G(\xi)\sin\theta(\xi)}{\bar{\xi} - \xi}\,\mathrm{d}\bar{\xi} \quad (39)$$

and

$$V \equiv \int_{\bar{\xi}=-1}^{0}\frac{G(\bar{\xi})\sin\theta(\bar{\xi})}{\bar{\xi}+1}\,\mathrm{d}\bar{\xi}. \quad (40)$$

Letting $U^A(\xi)$ and $U^B(\xi)$ be approximations by the Quad A and Quad B rules, each appropriately adjusted for the interval -1–0, gives

$$U^A(\xi) \cong \sum_{i=1}^{N} w_i^A \frac{G(\xi_i^A)\sin\theta(\xi_i^A) - G(\xi)\sin\theta(\xi)}{\xi_i^A - \xi} \quad (41)$$

and

$$U^B(\xi) \cong \sum_{i=0}^{N} w_i^B \frac{G(\xi_i^B)\sin\theta(\xi_i^B) - G(\xi)\sin\theta(\xi)}{\xi_i^B - \xi} \quad (42)$$

Similarly the V integral is approximated; only the Quad A rule is used to avoid a numerical difficulty at $\xi = -1$:

$$V^A \cong \sum_{i=1}^{N} w_i^A \frac{G(\xi_i^A)\sin\theta(\xi_i^A)}{\xi_i^A + 1}. \quad (43)$$

Now consider the two sets of integration points ξ_i^A and ξ_i^B that interlace in the interval -1–0 in some manner to form a set of $2N + 1$ points. The angle $\theta(\xi)$ evaluated at these points provides $2N + 1$ unknown values. Since the isotherm $T = 1$ in Fig. 1(b) must be normal to the insulated walls 1–2 and 3–4, the $\theta(\xi = -1) = \theta(\xi = 0) = 0$ and $2N - 1$ unknowns remain.

Referring now to (36), $N - 1$ "A" equations are constructed by approximations of the $U(\xi)$ integral by the Quad A rule evaluated at the ξ_i^B points, and N "B" equations are formed using the Quad B rule at the ξ_i^A points. This yields a system of $2N - 1$ transcendental equations for the $\theta(\xi)$:

$$G(\xi_j^B)\left[\cos\theta(\xi_j^B) - \frac{1}{\pi}\sin\theta(\xi_j^B)\ln\left(\frac{\xi_j^B + 1}{-\xi_j^B}\right)\right] = G(\xi = -1) - \frac{1}{\pi}U^A(\xi_j^B) + \frac{1}{\pi}V^A,$$

$$j = 1, 2, \ldots, N-1 \quad (44a)$$

$$G(\xi_j^A)\left[\cos\theta(\xi_j^A) - \frac{1}{\pi}\sin\theta(\xi_j^A)\ln\left(\frac{\xi_j^A + 1}{-\xi_j^A}\right)\right] = G(\xi = -1) - \frac{1}{\pi}U^B(\xi_j^A) + \frac{1}{\pi}V^A,$$

$$j = 1, 2, \ldots, N \quad (44b)$$

where $U^A(\xi_j^B)$ is directly from (41) with $\xi = \xi_j^B$, and from (42)

$$U^B(\xi_j^A) = w_0^B \frac{G(\xi_j^A)\sin\theta(\xi_j^A)}{1 + \xi_j^A} + w_N^B\frac{G(\xi_j^A)\sin\theta(\xi_j^A)}{\xi_j^A} + \sum_{i=1}^{N-1} w_i^B \frac{G(\xi_i^B)\sin\theta(\xi_i^B) - G(\xi_j^A)\sin\theta(\xi_j^A)}{\xi_i^B - \xi_j^A} \quad (44c)$$

since $\theta(\xi_0^B) = \theta(\xi_N^B) = 0$, and $\xi_0^B = -1$, $\xi_N^B = 0$.

For the solutions here, two relatively simple integration schemes were used. The Quad B rule was the extended trapezoid rule:

$$w_i^{\mathrm{B}} = \begin{cases} \dfrac{1}{2} & i = 0 \text{ or } N, \\ 1 & i = 1, 2, \ldots, N-1. \end{cases} \qquad x_i^{\mathrm{B}} = \frac{i}{N} - 1, \quad i = 0, 1, 2, \ldots, N.$$

For Quad A the scheme was

$$w_i^{\mathrm{A}} = 1, \qquad x_i^{\mathrm{A}} = \frac{i - \dfrac{1}{2}}{N} - 1,$$

$$i = 1, 2, \ldots, N. \qquad i = 1, 2, \ldots, N.$$

While convergence for these rules may not be as rapid as for some others that could have been employed, they have the advantage of producing together a set of $2N + 1$ evenly spaced points. This was useful for subsequent numerical operations in the solution procedure.

Solutions of the resulting non-linear system (44) were obtained through the minimization of the corresponding sum of squares function,

$$\sum_{j=1}^{2N-1} R_j^2$$

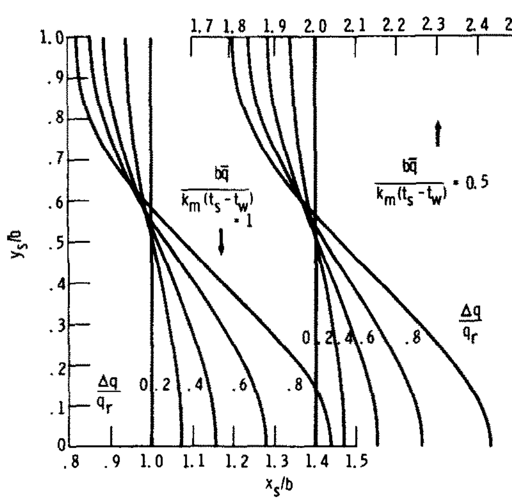
where the residuals R_j are the difference between the two sides of equation (44a) and similarly for (44b). The minimization was carried out by using a non-linear search routine employing the conjugate gradient method [10]. The Newton–Raphson technique could also have been applied. All necessary derivatives were numerically calculated and good convergence was obtained.

RESULTS AND DISCUSSION

To illustrate the use of the method, and obtain information on how the solidification interface responds to spatial heating variations, a cosine variation was selected with half-wavelength b,

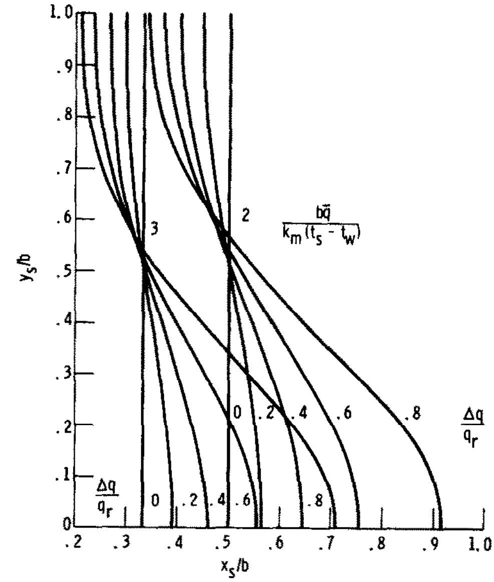
$$\frac{q_s}{q_r} = G = 1 - \frac{\Delta q}{q_r}\cos\left(\pi\frac{y}{b}\right) = 1 - \frac{\Delta q}{q_r}\cos\left(\pi\frac{Y}{B}\right). \tag{45}$$

Using the procedure outlined earlier, the interface coordinates were obtained and are shown in Fig. 4. Various $\Delta q/q_r$ curves are given for each B. As B is increased, the solid region becomes thin in the x-direction relative to its height (note that various abscissa scales are used).



(a) DIMENSIONLESS HEIGHTS, B = 0.5, 1.0.

FIG. 4(a). Interface shapes for various amplitudes of heat flux variation along interface.



(b) DIMENSIONLESS HEIGHTS, B = 2, 3.

FIG. 4(b). Interface shapes for various amplitudes of heat flux variation along interface.

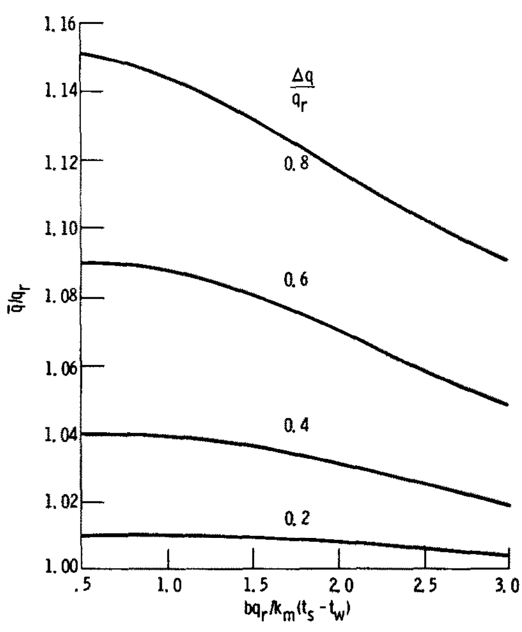


FIG. 5. Ratio of average to reference heat flux as a function of physical parameters.

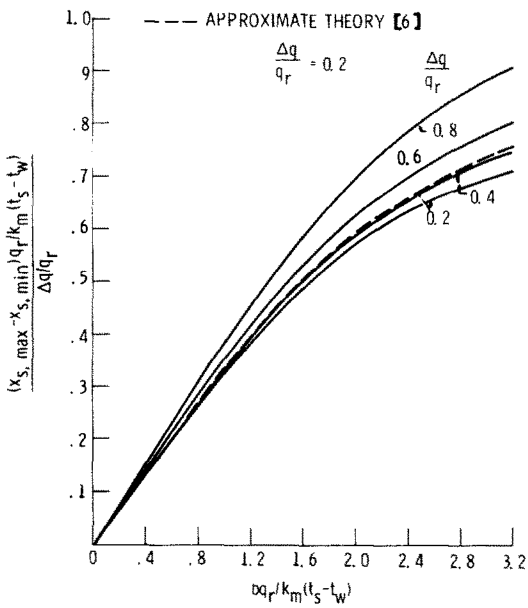


FIG. 6. Amplitude of interface distortion relative to flux variation at interface.

Because $B \equiv b\bar{q}/k_m(t_s - t_w)$, for a fixed b and $k_m(t_s - t_w)$, all the solid shapes in Fig. 4 for each B have the same total heat flow $Q = \bar{q}b$. However the total heat flow is not known *a priori*, so the curves in Fig. 4 must be used in conjunction with another set of results relating $\bar{q}$ to the known q_r. The ratio $\bar{q}/q_r$ was obtained in the analysis, and when this is divided into B it yields the parameter $q_r b/k_m(t_s - t_w)$ which is in terms of known quantities, and is the abscissa of Fig. 5. To use the results this figure would be used first to find $\bar{q}/q_r$ from the physical quantities. Then B can be calculated, and the appropriate interface shape interpolated from Fig. 4. For a plane interface ($\Delta q = 0$) the total heat flow through the region is bq_r. When $\Delta q \neq 0$, the total heat flow is $b\bar{q}$. Hence the ordinate of Fig. 5 gives the increase of total heat flow that occurs when the interface becomes curved.

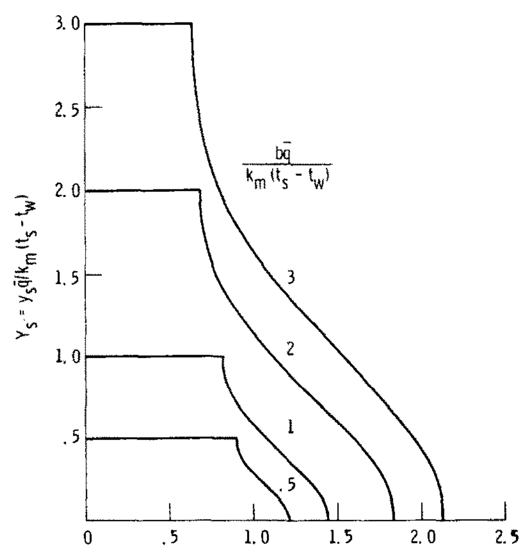


FIG. 7. Interface shapes for $\Delta q/q_r = 0.8$ and various dimensionless heights.

The interface shapes are of a cosine type shape, the thinner region at $y_s/b = 1$ corresponding to the location of largest imposed heating as a lower resistance to heat flow is needed to pass the imposed heating through the solid to the cold wall. The solid thickness tends to be inverse to the imposed heating. A quantity of interest is the amplitude response of the solidification interface to the amplitude of the imposed cosine heating. For $\Delta q = 0$ the uniform thickness is $x_s|_{\Delta q=0} = k_m(t_s - t_w)/q_r$. For $\Delta q \neq 0$ the local thickness is $x_s|_{\Delta q \neq 0} = X_s[k_m(t_s - t_w)/\bar{q}]$. Then the relative change in x_s ratioed to the change of q is

$$\frac{(x_{s,\max} - x_{s,\min})_{\Delta q \neq 0}/(x_s)_{\Delta q=0}}{(q_{s,\max} - q_{s,\min})_{\Delta q \neq 0}/(q_s)_{\Delta q=0}} = \frac{(q_r/\bar{q})(X_{s,\max} - X_{s,\min})}{2\Delta q/q_r}$$

which can be readily calculated from the results of Figs. 4 and 5. The amplitude response is given in Fig. 6. Also shown for $\Delta q/q_r = 0.2$ are results from the small amplitude theory in [6], and good agreement is obtained. The amplitude response is shown to be a minor function of $\Delta q/q_r$, but increases with $q_r b/k_m(t_s - t_w)$. For the larger values of this parameter the layer height becomes large relative to its thickness and the heat flow tends to be locally 1-dim.

The amplitude response and the nature of the solidified shapes are further illustrated by Fig. 7 where X_s and Y_s are plotted. For uniform heating, $\Delta q = 0$, the $X_s = 1$ for all Y_s. Hence these regions are all of approximately the same average thickness and the effect on shape is shown as the height is increased.

The results shown correspond to the cosine heating in equation (45). Interactive solutions could be made by combining the method with an analysis of heat flow in the liquid phase. The procedure here offers a partial alternative to a fully numerical solution and provides some results for comparison with that type of solution.

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APPENDIX
MAPPING BETWEEN u AND W-PLANES

Using the Schwarz–Christoffel transformation

$$W = C_1 \int \frac{du}{\sqrt{u+1}\sqrt{u}\sqrt{u-d}} + C_2. \tag{A.1}$$

Between points 2 and 1 in Figs. 2 and 3,

$$W_2 - W_1 = -1 = \frac{C_1}{-i}\int_{\xi=-\infty}^{-1} \frac{d\xi}{\sqrt{-\xi-1}\sqrt{-\xi}\sqrt{-\xi+d}}.$$

Using relation 231.00 in [8]

$$-1 = \frac{C_1}{-i}\frac{2}{\sqrt{d+1}}F\left(\frac{\pi}{2}, k'\right) = \frac{C_1}{-i}\frac{2}{\sqrt{d+1}}K(k') \tag{A.2}$$

where

$$k' = \sqrt{\frac{d}{d+1}}.$$

Between points 3 and 2

$$W_3 - W_2 = -i\tilde{Q} = \frac{C_1}{-1}\int_{\xi=-1}^{0} \frac{d\xi}{\sqrt{\xi+1}\sqrt{-\xi}\sqrt{-\xi+d}}.$$

Using relation 233.00 in [8], gives

$$i\tilde{Q} = C_1 \frac{2}{\sqrt{d+1}}K(k) \quad \text{where } k = \frac{1}{\sqrt{d+1}}.$$

Use equation (A.2) to eliminate C_1 and obtain

$$\tilde{Q} = \frac{K(k)}{K(k')}. \tag{A.3}$$

Equation (14) is derived from (A.1) by using (A.2) to eliminate C_1, and $k = (d+1)^{-1/2}$ to eliminate d.

METHODE INTEGRALE DE CAUCHY POUR LES FORMES D'INTERFACES DE SOLIDIFICATION BIDIMENSIONNELLE

Résumé—On développe une méthode pour déterminer la forme des interfaces de solidification en régime permanent, lorsque le liquide circule sur une surface froide. L'interface de solidification qui est à température uniforme prend une forme telle que l'énergie non uniforme convectée vers elle est localement équilibrée par la conduction dans le solide. La forme de l'interface est intéressante pour la structure du cristal pendant la solidification; ce qui a une application dans le forgeage des composites métalliques. Les résultats se transposent aux cas de stockage d'énergie par changement de phase, quand on a besoin de la configuration solidifiée et du transfert thermique global. L'analyse utilise une technique de transformation conforme liée aux coordonnées de l'interface désirée et aux composantes du gradient de température à l'interface. Ces composantes sont inconnues parce que la forme de l'interface est inconnue. Une formulation intégrale de Cauchy fournit une seconde relation donnant les composantes et une solution simultanée fournit la forme de l'interface.

EIN INTEGRALVERFAHREN NACH CAUCHY ZUR BERECHNUNG DER GESTALT VON ZWEIDIMENSIONALEN ERSTARRUNGS-GRENZFLÄCHEN

Zusammenfassung—Ein Verfahren wurde entwickelt, um die Gestalt einer stationären Erstarrungsgrenzfläche zu bestimmen, die sich bildet, wenn eine Flüssigkeit mit einer Temperatur oberhalb ihres Gefrierpunktes über eine kalte Oberfläche strömt. Die Grenzfläche, die eine einheitliche Temperatur hat, wird eine solche Form annehmen, daß die durch Konvektion ungleichförmig zugeführte Energie im Gleichgewicht mit der durch Leitung an den Feststoff abgegebenen Energie steht. Die Form der Grenzfläche ist im Hinblick auf die Kristallstruktur interessant, die sich während der Erstarrung ausbildet; die Beeinflussung der Kristallstruktur findet Anwendung beim Gießen natürlich verstärkter metallischer Kompositwerkstoffe. Die Ergebnisse betreffen auch Latentwärmespeicher-Konstruktionen, wo die Verfestigungs-Konfiguration und der Gesamtwärmeübergang benötigt werden. Die Berechnung erfolgt nach einem Verfahren der konformen Abbildung, womit die Beziehung zwischen den gesuchten Grenzflächenkoordinaten und den Komponenten des Temperaturgradienten an der Grenzfläche hergestellt wird. Diese Komponenten sind unbekannt, da die Grenzflächenform unbekannt ist. Eine Integralformulierung nach Cauchy liefert eine zweite Beziehung für die Komponenten, und über eine simultane Lösung wird die Form der Grenzfläche berechnet.

ИСПОЛЬЗОВАНИЕ ИНТЕГРАЛЬНОГО МЕТОДА КОШИ ДЛЯ ОПРЕДЕЛЕНИЯ ПРОФИЛЕЙ ДВУМЕРНЫХ ГРАНИЦ ЗАТВЕРДЕВАНИЯ

Аннотация — Разработан метод определения формы границ твердой фазы, образующейся в процессе стационарного затвердевания, когда жидкость с температурой выше точки замерзания циркулирует вдоль охлажденной поверхности. Граница затвердевания, имеющая постоянную температуру, приобретает такую форму, при которой энергия, подводимая к ней конвекцией, локально уравновешивается передачей тепла теплопроводностью в твердую фазу. Форма границы раздела представляет интерес в связи с исследованием кристаллических структур, образующихся при затвердевании; управление процессом образования кристаллической структуры имеет значение при отливке естественно упрочненных металлических композиций. Результаты представляют также интерес в связи с разработкой аккумуляторов энергии с использованием фазового перехода, когда необходимы сведения о конфигурации твердой фазы и величине суммарного теплопереноса. При анализе используется метод конформного отображения, с помощью которого искомые координаты границы раздела связываются с компонентами температурного градиента на поверхности раздела. Эти компоненты неизвестны, так как неизвестна форма границы раздела. Интегральным методом Коши получено дополнительное соотношение для этих компонент, что дает возможность определить форму границы раздела.